Signal Analysis & Communication ECE 355

Ch. 3.3: Fourier Series for CT Periodic Signals.
(Contd.)

Lecture 15

12-10-2023

Ch. 3.3: Fourier Series for CT Periodic Signals (Contd.)

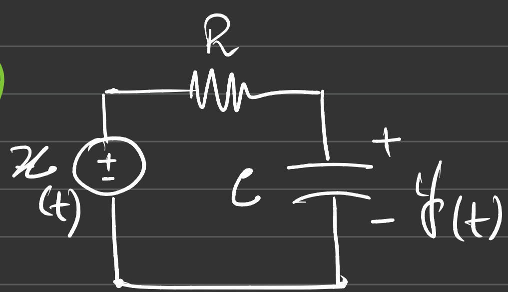
Recall: "Almost all" periodic signals can be expressed as a sum of harmonically related complex exponentials.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

LTI: $x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \sum_{k=-\infty}^{\infty} a_k \underline{H(jk\omega_0)} e^{jk\omega_0 t}$

Needs to be found!

Example:
(Low pass filter)



$H(s) = ?$

[If $x(t)$ is periodic, we need $H(jk\omega_0)$]

- Write LCCDE

$$RC \frac{dy(t)}{dt} + y(t) = x(t) \quad -①$$

(assume initial rest condition)

- To find $H(s)$, there can be two ways.

Way I.

i) Solving LCCDE for an arbitrary I/P with zero initial condition. (via the method of integrating factors)

$$y(t) = \int_{-\infty}^{t} \frac{1}{RC} e^{-(t-\tau)/RC} x(\tau)\, d\tau.$$

Try it yourselves!

ii) With $x(t) = \delta(t)$, $y(t) = h(t)$

$$h(t) = \int_{-\infty}^{t} \frac{1}{RC} e^{-(t-\tau)/RC} \underbrace{\delta(\tau)\, d\tau}_{\text{exists at } \tau = 0} = \frac{1}{RC} e^{-t/RC} u(t)$$

iii) $$H_{(s)} = \int_{-\infty}^{\infty} h_{(t)} e^{-st} dt$$

Way II

i) Use $x_{(t)} = e^{st}$ (consider complex exp. I/P)

$\Rightarrow y_{(t)} = H_{(s)} e^{st}$ (derived in lecture 13)

ii) Insert in eqn ① to find $H_{(s)}$

eqn ① $\Rightarrow RC H_{(s)} s e^{st} + H_{(s)} e^{st} = e^{st}$

$$H_{(s)} = \frac{1}{1+RCs}$$

$y_{(t)} = H_{(s)} e^{st}$

✓ Way II is easy!

- If we want to see the freq. response $(s = j\omega)$ — later

$$H_{(j\omega)} = \frac{1}{1+jRC\omega}$$

- And for periodic signals (expressed as sum of harmonically related sinusoids)

$$H_{(jk\omega_0)} = \frac{1}{1+jkRC\omega_0} \quad - ②$$

Remember
$y_{(t)} = \sum a_k H_{(s)_k} e^{s_k t}$

$s_k = k\omega_0$

- Suppose $RC = 1$ (for simplicity)

& $x_{(t)} = 1 + \cos(2\pi t) + \cos(4\pi t) + \cos(6\pi t)$ — ③

Find $y_{(t)}$

- I. $\omega_0 = 2\pi$ fundamental period.

II. $x_{(t)} = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kt}$ (Fourier Series) where $a_0 = 1$

$$a_1 = a_{-1} = a_2 = a_{-2} = a_3 = a_{-3} = \frac{1}{2}$$

& $a_k = 0$ for $|k| > 3$

From eqn ③

iii. Inserting in ②

$$H_{(jk\omega_0)} = \frac{1}{1+j2\pi k} \overset{*}{=} \frac{1}{\sqrt{1+4\pi^2k^2}} e^{-j\tan^{-1}(2\pi k)}$$

* Recall: $a + jb = \sqrt{a^2+b^2}\, e^{j\tan^{-1}(b/a)}$

$$\left|\frac{1}{z}\right| = \frac{1}{|z|} \; ; \quad \arg\frac{1}{z} = -\arg z$$

iv. $$y_{(t)} = \sum_{k=-\infty}^{\infty} a_k H_{(jk\omega_0)} e^{jk\omega_0 t}$$ (Lecture 13)

$$= \sum_{k=-\infty}^{\infty} a_k \frac{1}{\sqrt{1+4\pi^2k^2}} e^{j(2\pi kt - \tan^{-1}(2\pi k))}$$

$$\overset{*}{=} a_0 + \sum_{k=1}^{3} 2a_k \frac{1}{\sqrt{1+4\pi^2k^2}} \cos\left(2\pi kt - \tan^{-1}(2\pi k)\right)$$

* Recall $\cos\theta = \dfrac{e^{j\theta}+e^{-j\theta}}{2}$

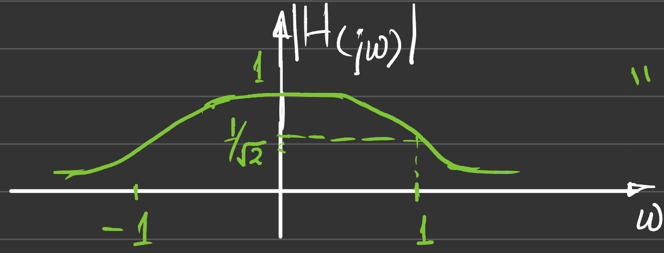
$$y_{(t)} = 1 + \frac{1}{\sqrt{1+4\pi^2}} \cos(\underbrace{2\pi}_{\omega_0} t - \tan^{-1}(2\pi))$$
$$+ \frac{1}{\sqrt{1+16\pi^2}} \cos(\underbrace{4\pi}_{\omega_1} t - \tan^{-1}(4\pi))$$
$$+ \frac{1}{\sqrt{1+36\pi^2}} \cos(\underbrace{6\pi}_{\omega_2} t - \tan^{-1}(6\pi))$$

Freq. dependent attenuation

Freq. dependent phase shift

NOTE 1

$$|H(j\omega)| \downarrow = \left|\frac{1}{1+j\omega}\right| = \frac{1}{\sqrt{1+\omega^2}} \uparrow \quad - (4)$$

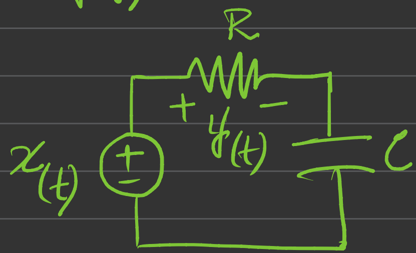


"Low Pass Filter" (LPF)

The system with freq. response $H(j\omega)$ in eqn (4) passes low frequencies & stops high frequencies.

NOTE 2

If the o/p $y(t)$ is across R in RC Circuit



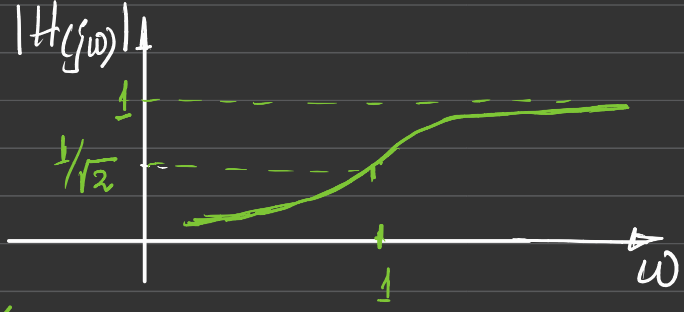
Then,

$$H(j\omega) = 1 - \frac{1}{1+j\omega} = \frac{j\omega}{1+j\omega}$$

$$|H(j\omega)| = \frac{\omega}{\sqrt{1+\omega^2}} \quad - (5)$$

"High Pass Filter" (HPF)

The sys. with freq. response $|H(j\omega)|$ in eqn (5) passes high frequencies & stops low frequencies.



CONCLUSION: Simple RC Circuit can be a LPF or HPF depending on the output across 'C' or 'R'!

CALCULATION OF CT FOURIER SERIES:

THEOREM

If $x_{(t)}$ has a Fourier series representation

$$x_{(t)} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \text{— (A)}$$

then

$$a_k = \frac{1}{T} \underbrace{\int_T}_{} x_{(t)} e^{-jk\omega_0 t} dt \quad \text{— (B)}$$

integration over any period T, i.e., $0 - T$, $-T/2 - T/2$ etc.

Proof

RHS of (B)

$$= \frac{1}{T} \int_T x_{(t)} e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T} \int_T \sum_{m=-\infty}^{\infty} a_m e^{jm\omega_0 t} \cdot e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T} \sum_{m=-\infty}^{\infty} a_m \underbrace{\int_T e^{j(m-k)\omega_0 t} dt}_{}$$

$$= \begin{cases} 0 & m \neq k \\ T & m = k \end{cases} = T\delta_{(m-k)}$$

Remember $\omega_0 = 2\pi/T$

$$= \frac{1}{T} a_k T = a_k$$

$$= \text{LHS of (B)}$$

NOTE

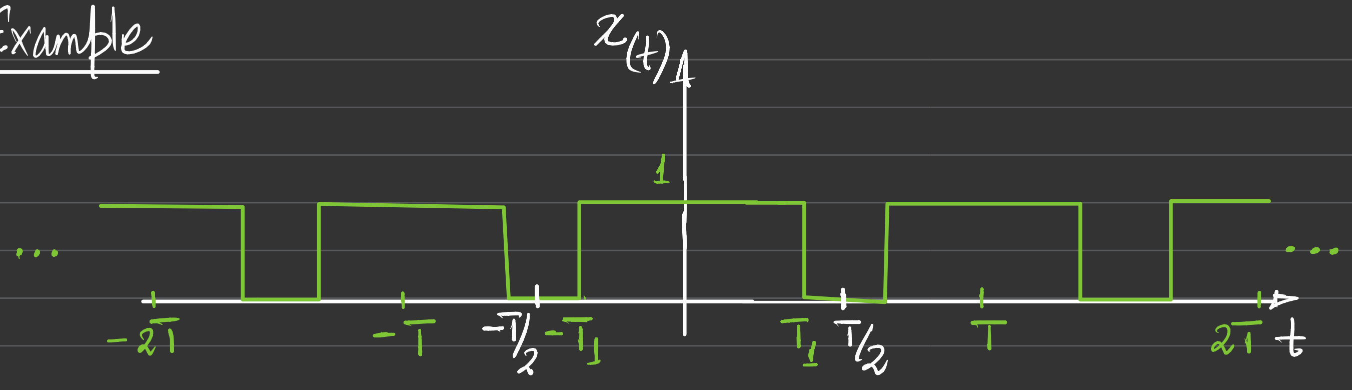
① Eqn (A) is called Synthesis Equation of FS

Eqn (B) is called Analysis Equation of FS

② $\{a_k\}$ are called FS coefficients.

$a_0 = \frac{1}{T} \int_T x_{(t)} dt$ is the average value of $x_{(t)}$

Example



$T > 2T_1$

$$x_{(t)} = \sum_{k=-\infty}^{\infty} p_{T_1}(t + kT)$$

where

$$p_{T_1} = \begin{cases} 1 & |t| < T_1 \\ 0 & o/w \end{cases}$$

Find Fourier Series representation of $x_{(t)}$.

$$x_{(t)} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \text{— (C)}$$

I. Fundamental freq. $= \omega_0 = \dfrac{2\pi}{T}$

II. $a_0 = \dfrac{1}{T}\displaystyle\int_{-T_1}^{T_1} 1 \cdot dt = \dfrac{2T_1}{T}$ — (D) We consider period from $-T/2$ to $T/2$

III. $a_k = \dfrac{1}{T}\displaystyle\int_{-T_1}^{T_1} e^{-jk\omega_0 t}$

$$= \frac{1}{-jk\omega_0 T} e^{-jk\omega_0 t} \Big|_{-T_1}^{T_1}$$

$$= \frac{\sin(k\omega_0 T_1)}{k\pi} \quad \text{— (E)}$$

Case

Let $T = 4T_1$

$$\Rightarrow \omega_0 = \frac{2\pi}{4T_1} = \frac{\pi}{T_1}$$

Ⓔ

$$a_k = \begin{cases} \dfrac{\sin(k\pi/2)}{k\pi}, & k \neq 0 \\ 1/2, & k = 0 \end{cases}$$

Ⓓ

$a_k = 0$ for k: even

$a_k \neq 0$ for k: odd

Overall an even function $a_k = a_{-k}$

- Therefore, Ⓒ for this case $\Rightarrow$

$$x_{(t)} \overset{*}{=} \frac{1}{2} + \frac{2}{\pi}\cos\left(\frac{2\pi}{T}t\right) - \frac{2}{3\pi}\cos\left(\frac{6\pi}{T}t\right) + \frac{2}{5\pi}\cos\left(\frac{10\pi}{T}t\right) - \dots$$

* again applying Euler's identity $\cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$

We'll see later!

So basically, whenever the sig. is even, we use

$$x_{(t)} = a_0 + \sum_{k=1}^{\infty} 2a_k \cos(k\omega_0 t)$$

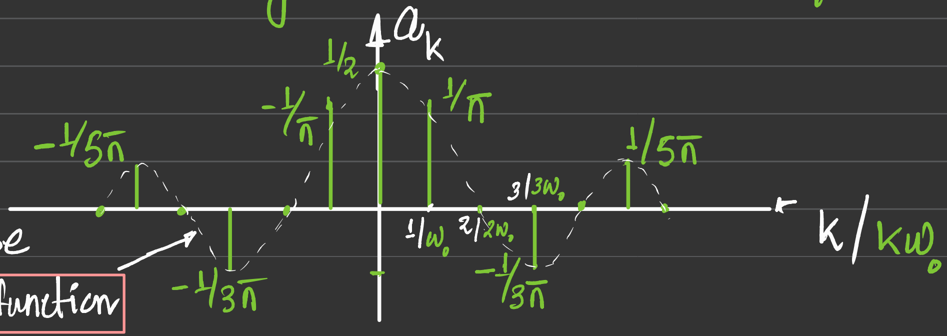
& whenever the sig. is odd, we use.

$$x_{(t)} = \sum_{k=1}^{\infty} 2a_k \sin(k\omega_0 t)$$

using the fact $\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{j2}$

- Plotting a_k (freq. domain representation of $x_{(t)}$).

[Representing which freq. components are present in the sig. & with how much amplitude]



v. imp. (later) { Envelope / $\boxed{\text{sine function}}$